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Separation Axioms for Completeness and Total Boundedness in Fuzzy Pseudo Metric Spaces

ZIKE DENG

Department of Mathematics, Hunan University, Changsha, Hunan, People's Republic of China

Submitted by L. A. Zadeh

This paper treats separation axioms, Baire theorem, contraction mapping theorem, and total boundedness in fuzzy pseudo metric spaces. © 1985 Academic Press, Inc.

1. INTRODUCTION

This paper is a continuing study of [3]. Separation axioms are introduced and investigated. The fuzzy unit interval used here is a little different from that one in [2] but Urysohn's lemma still holds. Baire theorem and contraction mapping theorem are proved in fuzzy pseudo metric spaces. Total boundedness is introduced and its relation to m-compactness is treated.

2. PRELIMINARIES

DEFINITION 2.1. $\mu \in I^X$, where $I = [0, 1]$ is called a fuzzy set on X.

$$\mu_1 \prec \mu_2 \Leftrightarrow \mu_2(x) \geq \mu_1(x) \qquad \forall x \in X.$$

$$\left(\bigvee_\alpha \mu_\alpha\right)(x) = \sup_\alpha \{\mu_\alpha(x)\} \qquad \forall x \in X.$$

$$\left(\bigwedge_\alpha \mu_\alpha\right)(x) = \inf_\alpha \{\mu_\alpha(x)\} \qquad \forall x \in X.$$

$$\tilde{1}(x) = 1 \qquad \forall x \in X.$$

$$\tilde{0}(x) = 0 \qquad \forall x \in X.$$

$$\mu^c(x) = 1 - \mu(x) \qquad \forall x \in X.$$

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DEFINITION 2.2. Let $f, X \to Y, \mu \in I^X, \nu \in I^Y$,

$$f(\mu)(y) = \sup_{x \in f^{-1}(y)} \{\mu(x)\}, \qquad f^{-1}(y) \neq \varnothing.$$
$$= 0, \qquad f^{-1}(y) = \varnothing.$$
$$f^{-1}(\nu)(x) = \nu(f(x)) \qquad \forall x \in X.$$

DEFINITION 2.3. $q_x^\alpha \in I^X$ $(0 < \alpha < 1)$, where

$$q(x') = \alpha, \qquad x' = x,$$
$$= 0, \qquad x' \neq x,$$

is called a fuzzy point on X (written q):

$$q^c = q_x^{1-\alpha}.$$
$$q \,\tilde{\in}\, \mu \Leftrightarrow \mu(x) > \alpha.$$

DEFINITION 2.4. $\mathscr{T} \subset I^X$, satisfying the following conditions:

(1) $\mu_\alpha \in \mathscr{T} \Rightarrow V\mu_\alpha \in \mathscr{T}$.
(2) $\mu_1 \in \mathscr{T}, \mu_2 \in \mathscr{T} \Rightarrow \mu_1 \wedge \mu_2 \in \mathscr{T}$.
(3) $\tilde{1} \in \mathscr{T}, \tilde{0} \in \mathscr{T}$.

$\mathscr{T}$ is called a fuzzy topology on X and $(X, \mathscr{T})$ fuzzy topological space.

DEFINITION 2.5. μ is called a neighbourhood of $q \Leftrightarrow \exists \nu \in \mathscr{T}$ $(q \,\tilde{\in}\, \nu \prec \mu)$. Let $\mathscr{N}_q$ denote the system of neighbourhoods of q.

DEFINITION 2.6. Let $d, \{q\} \times \{q\} \to [0, +\infty)$, satisfying the following conditions:

(I) $\alpha \leqslant \alpha_0 \Rightarrow d(q_{x_0}^\alpha q_{x_0}^{\alpha_0}) = 0$
(II) $d(q_1 q_2) = d(q_2^c q_1^c)$
(III) $d(q_1 q_3) \leqslant d(q_1 q_2) + d(q_2 q_3)$
(IV) $d(q_1 q_2) < r$, where $r > 0 \Rightarrow \exists \alpha_1' > \alpha_1 (d(q_{x_1}^{\alpha_1'} q_2) < r)$.

d is called a fuzzy pseudo metric on X and (X, d) fuzzy pseudo metric space.

We shall also use the following definition which is an equivalent version of Definition 2.6.

DEFINITION 2.7. Let d: $\{q\} \times \{q\} \to [0, +\infty)$, satisfying the following conditions:

(Ia) $\alpha \leqslant \alpha_0 \Rightarrow d(q_{x_0}^{\alpha} q_{x_0}^{\alpha_0}) = 0.$

(Ib) $\forall r > 0\ \exists \alpha' > \alpha_0\ (d(q_{x_0}^{\alpha'} q_{x_0}^{\alpha_0}) < r).$

(II) $d(q_1 q_2) = d(q_2^c q_1^c).$

(III) $d(q_1 q_3) \leqslant d(q_1 q_2) + d(q_2 q_3).$

d is called a fuzzy pseudo metric on X and (X, d) fuzzy pseudo metric space.

LEMMA 2.1. *Suppose the conditions* (Ia), (II), (III) *in Definition* 2.7 *hold, then the condition* (Ib) *is equivalent to the condition* (IV) *in Definition* 2.6.

Proof. (IV) $\Rightarrow$ (Ib) is immediate.

(Ib) $\Rightarrow$ (IV). If $d(q_1 q_2) < r$, where $r > 0$, then choose r' such that $0 < r' < r - d(q_1 q_2)$. By the condition (Ib), for this r' $\exists \alpha_1' > \alpha_1 (d(q_{x_1}^{\alpha_1'} q_1) < r')$. Thus $d(q_{x_1}^{\alpha_1'} q_2) \leqslant d(q_{x_1}^{\alpha_1'} q_1) + d(q_1 q_2) < r' + d(q_1 q_2) < r.$ Q.E.D.

DEFINITION 2.8. d is called a fuzzy metric on $X \Leftrightarrow d$ is a fuzzy pseudo metric and satisfies the condition that $d(q_1 q_2) = 0 \Rightarrow x_1 = x_2, \alpha_1 \leqslant \alpha_2$.

DEFINITION 2.9. q^n is said to converge to $q \Leftrightarrow \forall \mu \in \mathcal{N}_q\ \exists N$ $(n > N \Rightarrow q^n \tilde{\in} \mu)$ (written $q^n \to q$).

q^n is said to c-converge to $q \Leftrightarrow q^{nc} \to q^c$ (written $q^n \xrightarrow{c} q$).

q^n is said to m-converge to $q \Leftrightarrow q^n \to q$ and $q^n \xrightarrow{c} q$ (written $q^n \xrightarrow{m} q$).

DEFINITION 2.10. (X, d) is said to be m-compact $\Leftrightarrow$ every sequence q^n has an m-convergent subsequence.

DEFINITION 2.11. q^n is called a Cauchy sequence $\Leftrightarrow d(q^n q^m) \to 0$ $(n, m \to \infty)$.

DEFINITION 2.12. (X, d) is said to be complete $\Leftrightarrow$ every Cauchy sequence m-converges.

3. THE FUZZY UNIT INTERVAL

DEFINITION 3.1. Let $\lambda, R \to I$, where $R = (-\infty, +\infty)$ and $I = [0, 1]$. $\lambda \overset{*}{\searrow}$ on $I \Leftrightarrow \lambda$ is nonincreasing on I and $\lambda(t) \not\equiv \alpha$ $(0 < \alpha < 1)$ in any subinterval of I.

DEFINITION 3.2. The fuzzy unit interval is the set of all λ's which satisfy the condition that $\lambda \overset{*}{\searrow}$ on I and $\lambda(t) = 1\ \forall t < 0$, $\lambda(t) = 0\ \forall t > 1$ (written FI).

Note that λ_1 and λ_2 are considered to be identical iff $\lambda_1(t-)=\lambda_2(t-)$, $\lambda_1(t+)=\lambda_2(t+)$ $\forall t \in R$.

DEFINITION 3.3. $R_t\colon FI \to I$ and $R_t(\lambda)=\lambda(t+)$,
$L_t\colon FI \to I$ and $L_t(\lambda)=1-\lambda(t-)$.

DEFINITION 3.4. The fuzzy topology on FI is that which has $\{R_{t_1} \wedge L_{t_2}\colon t_1, t_2 \in (-\infty, +\infty)\}$ as a base.

DEFINITION 3.5. $\mu \overset{0}{\prec} \nu \Leftrightarrow \forall q\ (q \prec \mu \Rightarrow q \tilde{\in} \nu)$.

LEMMA 3.1. (1) $q \overset{0}{\prec} \mu \Leftrightarrow q \tilde{\in} \mu$.

(2) $\mu \overset{0}{\prec} \nu \Rightarrow f^{-1}(\mu) \prec f^{-1}(\nu)$.

(3) $R_t \prec L_t^c$.

(4) $t_1 > t_2 \Rightarrow L_{t_1}^c \overset{0}{\prec} R_{t_2}$.

Proof. (1) Is immediate.

(2) If $q \prec f^{-1}(\mu)$, then $f(q) \prec \mu$ (see [3], Theorem 3.2(6)), hence $f(q) \tilde{\in} \nu$ by hypothesis which implies $q \tilde{\in} f^{-1}(\nu)$ by [3, Theorem 3.2(5)].

(3) Is immediate.

(4) We have only to prove the case where $t_1, t_2 \in [0, 1]$, other cases are trivial. Let $q_\lambda^\alpha \prec L_{t_1}^c$, then $\lambda(t_1-)=L_{t_1}^c(\lambda) \geqslant \alpha$. Since $t_1 > t_2$ and λ is nonincreasing, $R_{t_2}(\lambda)=\lambda(t_2+) \geqslant \lambda(t_1-) \geqslant \alpha$. Thus $R_{t_2}(\lambda) \geqslant \alpha$. If $R_{t_2}(\lambda)=\alpha$, then $\forall t \in (t_2, t_1)$ we have $\alpha=\lambda(t_2+) \geqslant \lambda(t) \geqslant \lambda(t_1-) \geqslant \alpha$, hence $\lambda(t) \equiv \alpha$, a contradiction to $\lambda \overset{\bullet}{\curvearrowright}$ on I. Q.E.D.

4. SEPARATION AXIOMS

In this paragraph $(X, \mathscr{T})$ is a fuzzy topological space, $(q)^c$ denotes the complementary set of q.

DEFINITION 4.1. $(X, \mathscr{T})$ is said to be $T_1 \Leftrightarrow \forall q$ (q is closed).

DEFINITION 4.2. $(X, \mathscr{T})$ is said to be Hausdorff $\Leftrightarrow \forall q_1, q_2$ such that $q_1 \tilde{\in} (q_2)^c$ $\exists \mu_1, \mu_2 \in \mathscr{T}$ such that $\mu_1 \prec \mu_2^c$ ($q_1 \tilde{\in} \mu_1$ and $q_2 \tilde{\in} \mu_2$).

DEFINITION 4.3. $(X, \mathscr{T})$ is said to be regular $\Leftrightarrow \forall q\ \forall \nu \in \mathscr{T}$ such that $q \tilde{\in} \nu$ $\exists \mu \in \mathscr{T}$ $(q \tilde{\in} \mu \prec \bar{\mu} \overset{0}{\prec} \nu)$.

DEFINITION 4.4. $(X, \mathscr{T})$ is said to be normal $\Leftrightarrow \forall \mu^c \in \mathscr{T}\ \forall \nu \in \mathscr{T}$ such that $\mu \overset{0}{\prec} \nu$ $\exists \mu_1 \in \mathscr{T}$ $(\mu \overset{0}{\prec} \mu_1 \prec \bar{\mu}_1 \overset{0}{\prec} \nu)$.

Note that $\mu^c, \nu \in \mathscr{T}$, $\mu \overset{0}{\prec} \nu$ means every cluster point of μ is an interior point of ν (see [3, Theorems 7.1 and 7.6]).

DEFINITION 4.5. (1) $(X, \mathscr{T})$ is said to be $T_2 \Leftrightarrow$ it is Hausdorff.

(2) $(X, \mathscr{T})$ is said to be $T_3 \Leftrightarrow$ it is regular and T_1.

(3) $(X, \mathscr{T})$ is said to be $T_4 \Leftrightarrow$ it is normal and T_1.

THEOREM 4.1. (1) *$(X, \mathscr{T})$ is $T_4 \Rightarrow$ it is T_3.*

(2) *$(X, \mathscr{T})$ is $T_3 \Rightarrow$ it is T_2.*

(3) *$(X, \mathscr{T})$ is $T_2 \Rightarrow$ it is T_1.*

Proof. (1) If $q \tilde{\in} \nu \in \mathscr{T}$, then $(q)^c \in \mathscr{T}$ by T_1 and $q \overset{0}{\prec} \nu$ by Lemma 3.1(1). By normality $\exists \mu \in \mathscr{T}$ $(q \overset{0}{\prec} \mu \prec \bar{\mu} \overset{0}{\prec} \nu)$, and $q \overset{0}{\prec} \mu$ implies $q \tilde{\in} \mu$.

(2) If $q_1 \tilde{\in} (q_2)^c$, then $(q_2)^c \in \mathscr{T}$ by T_1 and $\exists \mu \in \mathscr{T}$ $(q_1 \tilde{\in} \mu \prec \bar{\mu} \overset{0}{\prec} (q_2)^c)$ by regularity. Evidently $\mu \prec (\bar{\mu}^c)^c$. And $\bar{\mu} \overset{0}{\prec} (q_2)^c$ implies $q_2 \overset{0}{\prec} \bar{\mu}^c$, which implies in turn $q_2 \tilde{\in} \bar{\mu}^c$ by Lemma 3.1(1). Take $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}^c$. Thus $\mu_1, \mu_2 \in \mathscr{T}$, $\mu_1 \prec \mu_2^c$, and $q_1 \tilde{\in} \mu_1$, $q_2 \tilde{\in} \mu_2$.

(3) If $q' \tilde{\in} (q)^c$, then by Hausdorff $\exists \mu_1, \mu_2 \in \mathscr{T}$ $(q' \tilde{\in} \mu_1, q \tilde{\in} \mu_2$ and $\mu_1 \prec \mu_2^c)$, hence $q' \tilde{\in} \mu_1 \prec \mu_2^c \prec (q)^c$. Thus $(q)^c \in \mathscr{T}$ by [3, Theorem 6.5]. Q.E.D.

THEOREM 4.2 (Urysohn's lemma). *$(X, \mathscr{T})$ is normal $\Leftrightarrow \forall \mu^c \in \mathscr{T}, \nu \in \mathscr{T}$ such that $\mu \overset{0}{\prec} \nu$ $\exists$ continuous $f: X \to FI$ $(\mu \prec f^{-1}(L_1^c)$ and $f^{-1}(R_0) \prec \nu)$.*

Proof. This proof is similar to that of [2, Theorem 1] with a few modifications only. We state it here for completeness.

($\Rightarrow$) Let $\mu^c \in \mathscr{T}$ and $\nu \in \mathscr{T}$ such that $\mu \overset{0}{\prec} \nu$. Then $\nu^c \overset{0}{\prec} \mu^c$. By the same process of construction as in [5, Sect. 11.1], for all rational r_n in I $\exists \mu_{r_n} \in \mathscr{T}$ $(\bar{\mu}_{r_i} \overset{0}{\prec} \mu_{r_j}$ for $r_i < r_j$ and $\nu^c \prec \bar{\mu}_{r_n} \prec \mu^c)$. Let $\nu_{r_n} = (\mu_{r_n}^c)^0$, then $\nu \succ \nu_{r_n} \succ \mu$ and $\nu_{r_i} \overset{0}{\succ} \bar{\nu}_{r_j}$ for $r_i < r_j$.

For $0 < t < 1$ define $\nu_t = \bigvee_{t < r_n} \nu_{r_n}$, then $\nu_t \in \mathscr{T}$ and $\nu \succ \nu_t \succ \mu$. If $t < t'$, choose r_{n_1}, r_{n_2} such that $t < r_{n_1} < r_{n_2} < t'$, then $\nu_t \succ \nu_{r_{n_1}}$ and $\nu_{r_{n_2}} \succ \nu_{t'}$, hence $\bar{\nu}_{t'} \prec \bar{\nu}_{r_{n_2}} \overset{0}{\prec} \nu_{r_{n_1}} \prec \nu_t$.

Define $f(x)(t) = \nu_t(x)$ for $0 < t < 1$. Since $\nu_{t'} \prec \bar{\nu}_{t'} \overset{0}{\prec} \nu_t$ for $t < t'$, $f(x) \in FI$. Evidently $\mu \prec \bigwedge_{t<1} \nu_t = f^{-1}(L_1^c)$ and $f^{-1}(R_0) = \bigvee_{t>0} \nu_t \prec \nu$. Since $f^{-1}(R_t) = \bigvee_{t'>t} \nu_{t'} \in \mathscr{T}$ and $f^{-1}(L_t^c) = \bigwedge_{t'<t} \nu_{t'} = \bigwedge_{t'<t} \bar{\nu}_{t'}$, which implies $f^{-1}(L_t) = (f^{-1}(L_t^c))^c = \bigvee_{t'<t} \bar{\nu}_{t'}^c \in \mathscr{T}$, f is continuous.

($\Leftarrow$) Let $\mu^c \in \mathscr{T}$ and $\nu \in \mathscr{T}$ such that $\mu \overset{0}{\prec} \nu$, then $\exists$ continuous $f: X \to FI$ $(\mu \prec f^{-1}(L_1^c)$ and $f^{-1}(R_0) \prec \nu)$. By Lemma 3.1(3), (4) for $0 < t < 1$, $L_1^c \overset{0}{\prec} R_t \prec L_t^c \overset{0}{\prec} R_0$, then $\mu \prec f^{-1}(L_t^c) \overset{0}{\prec} f^{-1}(R_t) \prec$

$f^{-1}(L_t^c) \overset{0}{\prec} f^{-1}(R_0) \prec \nu$ by Lemma 3.1(2). Take $\mu_1 = f^{-1}(R_t)$, then $f^{-1}(R_t) \in \mathcal{T}$ and $\overline{f^{-1}(R_t)} \prec \overline{f^{-1}(L_t^c)} = f^{-1}(L_t^c)$, for f is continuous. Thus $\mu \overset{0}{\prec} \mu_1 \prec \bar{\mu}_1 \overset{0}{\prec} \nu$. Q.E.D.

THEOREM 4.3. *(X, d) is a fuzzy metric pace $\Leftrightarrow$ it is a fuzzy pseudo metric space and T_1.*

Proof. ($\Rightarrow$) Given $q_{x_0}^{\alpha_0}$, let $\overline{q_{x_0}^{\alpha_0}}$ denote the closure of $q_{x_0}^{\alpha_0}$. If $q \prec \overline{q_{x_0}^{\alpha_0}}$, then q is a cluster point of $qq_{x_0}^{\alpha_0}$ by [3, Theorem 7.6]. Thus $\forall \varepsilon > 0\, \exists q' \tilde{\in} q_{x_0}^{\alpha_0}$ $(d(qq') < \varepsilon)$ by [3, Theorem 7.5]. Since $d(qq_{x_0}^{\alpha_0}) \leqslant d(qq')$ by [3, Theorem 4.4], $d(qq_{x_0}^{\alpha_0}) < \varepsilon\, \forall \varepsilon$, which implies $d(qq_{x_0}^{\alpha_0}) = 0$, which implies in turn $x = x_0, \alpha \leqslant \alpha_0$ by hypothesis, i.e., $q \prec q_{x_0}^{\alpha_0}$. It follows $\overline{q_{x_0}^{\alpha_0}} \prec q_{x_0}^{\alpha_0}$, hence $q_{x_0}^{\alpha_0} = \overline{q_{x_0}^{\alpha_0}}$, which implies $q_{x_0}^{\alpha_0}$ is closed.

($\Leftarrow$) Let $d(q_1, q_2) = 0$, then $\forall \varepsilon\ (d(q_1 q_2) < \varepsilon)$. By [3, Theorem 4.3] $\exists \alpha' < \alpha_2\ (d(q_1 q_{x_2}^{\alpha'}) < \varepsilon)$. It follows that q_1 is a cluster point of q_2 by [3, Theorem 7.5] which implies $q_1 \prec \bar{q}_2$ by [3, Theorem 7.6]. Since (X, d) is T_1, $q_2 = \bar{q}_2$; hence $x_1 = x_2, \alpha_1 \leqslant \alpha_2$. Q.E.D.

THEOREM 4.4. *A fuzzy pseudo metric space (X, d) is Hausdorff $\Leftrightarrow$ $(q^n \to q_1$ and $q^n \xrightarrow{c} q_2 \Rightarrow q_2 \prec q_1)$.*

Proof. ($\Rightarrow$) If $\exists \{q^n\}$ such that $q^n \to q_1, q^n \xrightarrow{c} q_2$ but $q_2 \not\prec q_1$, then $q_2 \not\prec q_1 \Rightarrow q_2^c \tilde{\in} (q_1)^c$ and $q^n \to q_1$ and $q^n \xrightarrow{c} q_2 \Rightarrow \forall \mu_1, \mu_2 \in \mathcal{T}$ such that $q_1 \tilde{\in} \mu_1, q_2^c \tilde{\in} \mu_2\, \exists n_0\, (q^{n_0} \tilde{\in} \mu_1, q^{n_0 c} \tilde{\in} \mu_2)$. But $q^{n_0 c} \tilde{\in} \mu_2 \Rightarrow q^{n_0} \not\prec \mu_2^c$ and $q^{n_0} \tilde{\in} \mu_1$ and $q^{n_0} \not\prec \mu_2^c \Rightarrow \mu_1 \not\prec \mu_2^c$, a contradiction.

($\Leftarrow$) If not, then $\exists q_1, q_2$ such that $q_1 \tilde{\in} (q_2)^c$ and $\forall n (B(q_1(1/n)) \not\prec B^c(q_2(1/n)))$. Thus $\exists q^n \tilde{\in} B(q_1(1/n))$, but $q^n \not\prec B^c(q_2(1/n))$, which implies $q^{nc} \tilde{\in} B(q_2(1/n))$. Evidently $q^n \to q_1$ and $q^{nc} \to q_2$, which implies $q^n \xrightarrow{c} q_2^c$. But $q_1 \tilde{\in} (q_2)^c \Rightarrow q_2 \tilde{\in} (q_1)^c \Rightarrow q_2^c \not\prec q_1$, a contradiction. Q.E.D.

THEOREM 4.5. *A fuzzy metric space (X, d) is Hausdorff.*

Proof. If $q^n \to q_1, q^n \xrightarrow{c} q_2$, then $d(q_2 q_1) \leqslant d(q_2 q^n) + d(q^n q_1) = d(q^{nc} q_2^c) + d(q^n q_1)$, which implies $d(q_2 q_1) = 0$, which implies in turn $q_2 \prec q_1$ for d is a fuzzy metric. By Theorem 4.4 (X, d) is Hausdorff. Q.E.D.

THEOREM 4.6. *A fuzzy pseudo metric space (X, d) is normal.*

Proof. This proof is similar to that of [4, Theorem 5.7] with a few modifications only. We state it here for completeness.

Let $\mu^c \in \mathcal{T}$ and $\nu \in \mathcal{T}$ such that $\mu \overset{0}{\prec} \nu$. By [3, Theorems 7.3, 7.9], $\mu = \bigwedge_{r>0} D_r(\mu)$ and $\nu = \bigvee_Q \nu'$, where $Q = \{\nu', \exists r' (D_{r'}(\nu') \prec \nu)\}$. Let $\mu_1 = \bigvee_Q D_{r'/2}(\mu \wedge \nu')$.

By [3, Theorem 7.4] $\mu_1 \in \mathscr{T}$. Since $q \prec \mu \Rightarrow q \,\tilde{\in}\, \nu \Rightarrow \exists \nu' \in Q (q \,\tilde{\in}\, \nu')$, $q \prec \mu \wedge \nu'$. Thus $q \,\tilde{\in}\, D_{r'/2}(q) \prec D_{r'/2}(\mu \wedge \nu')$, which implies $q \,\tilde{\in}\, \mu_1$. It follows $\mu \overset{0}{\prec} \mu_1$.

To show that $\bar{\mu}_1 \overset{0}{\prec} \nu$ we need the following proposition:

$$q \prec \bar{\mu}_1 \Rightarrow \forall r > 0\ \exists \nu' \in Q\ (q \,\tilde{\in}\, D_r(\mu) \vee D_{r'}(\nu')). \qquad (*)$$

And we prove it as follows.

If $q \prec \bar{\mu}_1$ and $r > 0$ is given, then by [3, Theorem 7.5] for $r/2\ \exists q' \in \mu_1$ $(d(qq') < r/2)$. But $q' \in \mu_1 \Rightarrow \exists \nu' \in Q$ $(q' \,\tilde{\in}\, D_{r'/2}(\mu \wedge \nu')) \Rightarrow \exists q'' \,\tilde{\in}\, \mu \wedge \nu'$ $(d(q'q'') < r'/2)$. Thus $d(qq'') \leqslant d(qq') + d(q'q'') < (r + r')/2 \leqslant \max\{r, r'\}$. It follows that $q \,\tilde{\in}\, D_{\max\{r,r'\}}(\mu \wedge \nu') = D_r(\mu \wedge \nu') \vee D_{r'}(\mu \wedge \nu') \prec D_r(\mu) \vee D_{r'}(\nu')$.

Now let $q \prec \bar{\mu}_1$. If $\forall r > 0$ $(q \,\tilde{\in}\, D_r(\mu))$, then $q \prec \bigwedge_{r>0} D_r(\mu) = \mu \overset{0}{\prec} \nu$, hence $q \,\tilde{\in}\, \nu$. If $\exists r_0 > 0$ $(q \,\tilde{\notin}\, D_{r_0}(\mu))$, then by $(*)$, $q \,\tilde{\in}\, D_{r_0'}(\nu') \prec \nu$, hence $q \,\tilde{\in}\, \nu$. Thus $\bar{\mu}_1 \overset{0}{\prec} \nu$. Q.E.D.

DEFINITION 4.6. (X, d) is said to be separable $\Leftrightarrow \exists \{q^n\}$ such that $\forall q$ $\forall \varepsilon > 0\ \exists q^n$ $(q^n \,\tilde{\in}\, B(q\varepsilon)$ and $q^{nc} \,\tilde{\in}\, B(q^c\varepsilon))$.

DEFINITION 4.7. $(X, \mathscr{T})$ is said to be $C_{\text{II}} \Leftrightarrow \mathscr{T}$ has a countable base $(X, \mathscr{T})$ is said to be $C_{\text{I}} \Leftrightarrow$ every fuzzy point q has a countable base of system of neighbourhoods.

THEOREM 4.7. *A fuzzy pseudo metric space (X, d) is separable $\Leftrightarrow$ it is C_{II}.*

Proof. $(\Rightarrow)$ Let $\{q^n\}$ be such that $\forall q\, \forall \varepsilon > 0\ \exists q^n$ $(q^n \,\tilde{\in}\, B(q\varepsilon)$ and $q^{nc} \,\tilde{\in}\, B(q^c\varepsilon))$. We show that $\{B(q^n(1/k)) : n = 1, 2, \ldots, k = 1, 2, \ldots\}$ is countable base of $\mathscr{T}$ induced by d as follows:

Let $q \,\tilde{\in}\, \mu \in \mathscr{T}$, then $\exists B(q\varepsilon) \prec \mu$ by [3, Theorem 6.5]. Choose $1/k < \varepsilon/2$, then $\exists q^{n_0} \,\tilde{\in}\, B(q(1/k))$ and $q^{n_0c} \,\tilde{\in}\, B(q^c(1/k))$. Since $d(qq^{n_0}) = d(q^{n_0c}q^c) < 1/k$, $q \,\tilde{\in}\, B(q^{n_0}(1/k))$. Let $q' \,\tilde{\in}\, B(q^{n_0}(1/k))$, then $d(q'q) \leqslant d(q'q^{n_0}) + d(q^{n_0}q) < 2/k < \varepsilon$, $q' \,\tilde{\in}\, B(q\varepsilon)$, which implies $B(q^{n_0}(1/k)) \prec B(q\varepsilon) \prec \mu$.

$(\Leftarrow)$ Let $\{\mu_n\}$ be a countable base of $\mathscr{T}$ induced by d. Let $Q = \{\langle \mu_n, \mu_m \rangle : \mu_n \not\prec \mu_m^c\}$, $Q \neq \varnothing$, for $q \,\tilde{\in}\, \mu_n$ and $q^c \,\tilde{\in}\, \tilde{1} \Rightarrow \exists \mu_m$ $(q^c \,\tilde{\in}\, \mu_m) \Rightarrow \mu_n \not\prec \mu_m^c \Rightarrow \langle \mu_n, \mu_m \rangle \in Q$. Let q^{nm} be such that $q^{nm} \,\tilde{\in}\, \mu_n$ and $q^{nmc} \,\tilde{\in}\, \mu_m$, where $\langle \mu_n, \mu_m \rangle \in Q$.

Let q and $\varepsilon > 0$ be given. Then $\exists \mu_{n_1}$ $(q \,\tilde{\in}\, \mu_{n_1} \prec B(q\varepsilon))$ and $\exists \mu_{m_1}$ $(q^c \,\tilde{\in}\, \mu_{m_1} \prec B(q^c\varepsilon))$, for μ_n is a base. Thus $\mu_{n_1} \not\prec \mu_{m_1}^c$, which implies $\langle \mu_{n_1}, \mu_{m_1} \rangle \in Q$. It follows that $q^{nm} \,\tilde{\in}\, B(q\varepsilon)$ and $q^{nmc} \,\tilde{\in}\, B(q^c\varepsilon)$. Q.E.D.

THEOREM 4.8. *A fuzzy pseudo metric space (X, d) is C_{I}.*

Proof. By [3, Theorem 6.4, Corollary] $\{B(q(1/n)) : n = 1, 2, \ldots\}$ is a countable base of $\mathscr{N}_2$.

5. Completeness

Let (X, d) be a fuzzy pseudo metric space.

Definition 5.1. $S(q_0 r) = \bigvee_Q q$, where $Q = \{q\colon d(qq_0) \leqslant r\}$.

Lemma 5.1. (1) *$S(q_0 r)$ is closed.*

(2) $q \prec S(q_0 r) \Leftrightarrow d(qq_0) \leqslant r$.

(3) $\forall q \tilde{\in} \mu \in \mathscr{T}\ \exists S(qr) \overset{0}{\prec} \mu$.

Proof. (1) If $q \prec \overline{S(q_0 r)}$, then $\forall \varepsilon > 0\ \exists q' \tilde{\in} S(q_0 r)\ (d(qq') < \varepsilon)$ by [3, Theorem 7.5]. But $q' \tilde{\in} S(q_0 r) \Rightarrow \exists \tilde{q}\ (q' \tilde{\in} \tilde{q}$ and $d(\tilde{q}q_0) \leqslant r) \Rightarrow d(q'q_0) \leqslant d(\tilde{q}q_0) \leqslant r$. Thus $d(qq_0) \leqslant d(qq') + d(q'q_0) < r + \varepsilon$; $\underline{d(qq_0)} \leqslant r$, for ε is arbitrary. It follows that $q \prec S(q_0 r)$, which implies $\overline{S(q_0 r)} \prec S(q_0 r)$.

(2) $(\Leftarrow)$ Is immediate. $(\Rightarrow)$ If $q \prec S(q_0 r)$ then $q \prec \overline{S(q_0 r)}$ and by what has just been proved in (1), $d(qq_0) \leqslant r$.

(3) If $q \tilde{\in} \mu \in \mathscr{T}$, then $\exists B(qr) \prec \mu$. Choose $r' < r$, then $q' \prec S(qr')$ implies $d(q'q) \leqslant r' < r$ by (2), hence $q' \tilde{\in} B(qr) \prec \mu$. Thus $S(q_0 r') \overset{0}{\prec} \mu$. Q.E.D.

Definition 5.2. μ is said to be dense $\Leftrightarrow \forall q\ \forall \varepsilon > 0\ \exists q' \tilde{\in} \mu\ (q' \tilde{\in} B(q\varepsilon)$ and $q'^c \tilde{\in} B(q^c \varepsilon))$.

Theorem 5.1 (Baire). *(X, d) is a complete fuzzy pseudo metric space and $\mu_n \in \mathscr{T}$ induced by d is dense $\Rightarrow \bigwedge_1^\infty \mu_n$ is dense.*

Proof. Let q and $\varepsilon > 0$ be given. Since μ_1 is dense, $\exists q' \tilde{\in} \mu_1 (q' \tilde{\in} B(q\varepsilon)$ and $q'^c \tilde{\in} B(q^c \varepsilon))$. By Lemma 4.1(2) $\exists S(q'\varepsilon_1) \overset{0}{\prec} \mu_1 \wedge B(q\varepsilon)$, $S(q'^c \varepsilon_1) \overset{0}{\prec} B(q^c \varepsilon)$ and $\varepsilon_1 < \frac{1}{2}$. If we have constructed q^{n-1} such that $S(q^{n-1} \varepsilon_{n-1}) \overset{0}{\prec} \mu_{n-1} \wedge B(q^{n-2} \varepsilon_{n-2})$, $S(q^{n-1\,c} \varepsilon_{n-1}) \overset{0}{\prec} B(q^{n-2\,c} \varepsilon_{n-2})$ and $\varepsilon_{n-1} < 1/2^{n-1}$, then since μ_n is dense, $\exists S(q^n \varepsilon_n) \overset{0}{\prec} \mu_n \wedge B(q^{n-1} \varepsilon_{n-1})$, $S(q^{nc} \varepsilon_n) \overset{0}{\prec} B(q^{n-1\,c} \varepsilon_{n-1})$ and $\varepsilon_n < 1/2^n$.

If $n > m$, then $B(q^n \varepsilon_n) \prec B(q^m \varepsilon_m)$, hence $d(q^n q^m) < \varepsilon_m$. Likewise $d(q^m q^n) = d(q^{nc} q^{mc}) < \varepsilon_m$. It follows that $\{q^n\}$ is a Cauchy sequence. By completeness $\exists q^*\ (q^n \xrightarrow{m} q^*)$.

Since $d(q^* q') \leqslant d(q^* q^m) + d(q^m q') < d(q^* q^m) + \varepsilon_1$, $q^* \prec S(q'\varepsilon_1) \overset{0}{\prec} B(q\varepsilon)$, hence $q^* \tilde{\in} B(q\varepsilon)$. Likewise $q^{*c} \tilde{\in} B(q^c \varepsilon)$. On the other hand, $d(q^* q^n) \leqslant d(q^* q^m) + d(q^m q^n)$ for $m > n$, which implies $q^* \prec S(q^n \varepsilon_n) \prec \mu_n\ \forall n$. Thus $q^* \prec \bigwedge_1^\infty \mu_n$.

Finally since $q^{*c} \tilde{\in} B(q^c \varepsilon)$, $(q^{*c} \tilde{\in} \tilde{q}^c \tilde{\in} B(q^c \varepsilon))$. And $\tilde{q} \tilde{\in} q^* \tilde{\in} B(q\varepsilon)$, $\tilde{q} \tilde{\in} q^* \prec \bigwedge_1^\infty \mu_n$. Thus $\bigwedge_1^\infty \mu_n$ is dense. Q.E.D.

DEFINITION 5.3. Let $f: (X, d) \to (X, d)$, f is called a contraction $\Leftrightarrow f$ is continuous, and $d(f(q_1)\, f(q_2)) \leqslant \lambda d(q_1 q_2)$, where $0 < \lambda < 1$.

THEOREM 5.2. *(X, d) is a complete fuzzy metric space and f is a contraction $\Rightarrow \exists$ unique q_0 $(f(q_0) = q_0)$.*

Proof. For any q_0, let $q_0^1 = f(q_0)$, $q_0^2 = f(q_0^1), \ldots, q_0^n = f(q_0^{n-1}), \ldots$. Since f is a contraction, $d(q_0^{n+1} q_0^n) \leqslant \lambda^n d(f(q_0)\, q_0)$ and $d(q_0^n q_0^{n+1}) \leqslant \lambda^n d(q_0 f(q_0))$. Thus $d(q_0^{n+m+1} q_0^n) \leqslant \sum_{k=n}^{n+m} d(q_0^{k+1} q_0^k) \leqslant (\sum_{k=n}^{n+m} \lambda^k)\, d(f(q_0)\, q_0)$ and $d(q_0^n q_0^{n+m+1}) \leqslant (\sum_{k=n}^{n+m} \lambda^k)\, d(q_0 f(q_0))$. It follows that q_0^n is a Cauchy sequence and by completeness $\exists q^*$ $(q_0^n \xrightarrow{m} q^*)$. By continuity of f, $f(q_0^n) \xrightarrow{m} f(q^*)$. Thus $d(q_0 f(q_0)) \leqslant d(q_0 q_0^{n+1}) + d(q_0^{n+1} f(q_0^n)) + d(f(q_0^n)\, f(q_0))$ and $d(f(q_0)\, q_0) \leqslant d(f(q_0)\, f(q_0^n)) + d(f(q_0^n)\, q_0^{n+1}) + d(q_0^{n+1} q_0)$, which implies $d(q_0 f(q_0)) = d(f(q_0)\, q_0) = 0$, which implies in turn $q_0 \prec f(q_0)$ and $f(q_0) \prec q_0$, i.e., $f(q_0) = q_0$.

If there were q_1, q_2 such that $f(q_1) = q_1, f(q_2) = q_2$, then $d(q_1 q_2) = d(f(q_1)\, f(q_2)) \leqslant \lambda d(q_1 q_2)$, hence $d(q_1 q_2) = 0$. Likewise $d(q_2 q_1) = 0$, therefore $q_1 \prec q_2$ and $q_2 \prec q_1$, i.e., $q_1 = q_2$. Q.E.D.

6. TOTAL BOUNDEDNESS

Let (X, d) be a fuzzy pseudo metric space.

DEFINITION 6.1. (X, d) is said to be totally bounded $\Leftrightarrow \forall \varepsilon > 0$ $\exists \{q^n\}_{n=1,\ldots,N}$ $(\forall q\, \exists n\, (q \mathrel{\tilde{\in}} B(q^n \varepsilon)$ and $q^c \mathrel{\tilde{\in}} B(q^{nc}\varepsilon)))$.

THEOREM 6.1. *(X, d) is totally bounded $\Leftrightarrow \forall \{q^n\}\ \exists$ Cauchy subsequence $\{q^{n_i}\}$.*

Proof. $(\Rightarrow)$ By total boundedness $\forall n\, \exists q_j^n$, $j = 1, \ldots, m_n$, such that $\forall q\, \exists j$ $(q \mathrel{\tilde{\in}} B(q_j^n(1/n))$ and $q^c \mathrel{\tilde{\in}} B(q_j^{nc}(1/n)))$.

Let a sequence $\{p^i\}$ be given. For $n = 1$ $\exists$ subsequence $\{p_{(1)}^i\}$ of $\{p^i\}$ $\exists j$ $(p_{(1)}^i \mathrel{\tilde{\in}} B(q_j^1 1)$ and $p_{(1)}^{ic} \mathrel{\tilde{\in}} B(q_j^{1c} 1)\ \forall p_{(1)}^i)$. For $n = 2$ $\exists$ subsequence $\{p_{(2)}^i\}$ of $\{p_{(1)}^i\}$ $\exists j$ $(p_{(2)}^i \mathrel{\tilde{\in}} B(q_j^2 \tfrac{1}{2})$ and $p_{(2)}^{ic} \mathrel{\tilde{\in}} B(q_j^{2c} \tfrac{1}{2})\ \forall p_{(2)}^i)$. If $\{p_{(k-1)}^i\}$ has been constructed, then for $n = k$ $\exists$ subsequence $\{p_{(k)}^i\}$ of $\{p_{(k-1)}^i\}$ $\exists j$ $(p_{(k)}^i \mathrel{\tilde{\in}} B(q_j^k(1/k))$ and $p_{(k)}^{ic} \mathrel{\tilde{\in}} B(q_j^{kc}(1/k))\ \forall p_{(k)}^i)$.

Choose $p_{(k)}^k$, then for $k > i$, $d(p_{(i)}^i p_{(k)}^k) < 2/i$ and $d(p_{(k)}^k p_{(i)}^i) < 2/i$, hence $\{p_{(k)}^k\}$ is a Cauchy subsequence of $\{p^i\}$.

$(\Leftarrow)$ If not, then $\exists \varepsilon > 0$, for any $q^1\, \exists q^2$ $(q^2 \mathrel{\tilde{\notin}} B(q^1 \varepsilon)$ or $q^{2c} \mathrel{\tilde{\notin}} B(q^{1c}\varepsilon))$, which implies $d(q^2 q^1) \geqslant \varepsilon$ or $d(q^1 q^2) \geqslant \varepsilon$. For $q^1, q^2\, \exists q^3$ $(d(q^3 q^i) \geqslant \varepsilon$ or $d(q^i q^3) \geqslant \varepsilon\ \forall i = 1, 2)$. If $q^1 q^2 \cdots q^{n-1}$ has been constructed, then for $q^1 q^3 \cdots q^{n-1}\, \exists q^n$ $(d(q^n q^i) \geqslant \varepsilon$ or $d(q^i q^n) \geqslant \varepsilon\ \forall i = 1, 2, \ldots, n-1)$. Thus $\{q^n\}$ has the property that $d(q^nq^m) \geqslant \varepsilon$ or $d(q^mq^n) \geqslant \varepsilon \, \forall n \neq m$. Evidently it is impossible for $\{q^n\}$ to have a Cauchy subsequence. Q.E.D.

THEOREM 6.2. *(X, d) is m-compact $\Leftrightarrow$ it is totally bounded and complete.*

Proof. ($\Rightarrow$) If (X, d) were not totally bounded, then by the same process of construction as in Theorem 6.1 "$\Leftarrow$", $\exists \{q^n\}$ such that $d(q^nq^m) \geqslant \varepsilon$ or $d(q^mq^n) \geqslant \varepsilon \, \forall m \neq n$. By m-compactness, $\exists q^{n_i} \xrightarrow{m} q^*$, then $\exists i \neq j$ $(d(q^*q^{n_i}) < \varepsilon/2,$ $d(q^{n_i}q^*) < \varepsilon/2$ and $d(q^*q^{n_j}) < \varepsilon/2,$ $d(q^{n_j}q^*) < \varepsilon/2)$, which implies $d(q^{n_i}q^{n_j}) < \varepsilon$ and $d(q^{n_j}q^{n_i}) < \varepsilon$, a contradiction.

Let q^n be a Cauchy sequence. Then by m-compactness $\exists q^{n_i} \xrightarrow{m} q$. Since $d(q^nq) \leqslant d(q^nq^{n_i}) + d(q^{n_i}q)$ and $d(qq^n) \leqslant d(qq^{n_i}) + d(q^{n_i}q^n)$, $q^n \xrightarrow{m} q$.

($\Leftarrow$) Let $\{q^n\}$ be given. Then by Theorem 6.1 $\exists$ Cauchy subsequence $\{q^{n_i}\}$. By completeness $q^{n_i} \xrightarrow{m} q$. Q.E.D.

THEOREM 6.3. *(X, d) is totally bounded $\Rightarrow$ it is separable.*

Proof. $\forall n \, \exists q_1^n, q_2^n, \ldots, q_{N(n)}^n$ $(\forall q \, \exists i$ $(q \,\tilde{\in}\, B(q_i^n(1/n))$ and $q^c \,\tilde{\in}\, B(q_i^{nc}(1/n))))$. Consider $\{q_i^n, n = 1, 2, \ldots, i = 1, 2, \ldots, N(n)\}$ $\forall q \, \forall \varepsilon > 0$, choose $1/n < \varepsilon$, then $\exists i$ $(q \,\tilde{\in}\, B(q_i^n(1/n))$ and $q^c \,\tilde{\in}\, B(q_i^{nc}(1/n)))$. Thus $d(qq_i^n) < \varepsilon$ and $d(q_i^nq) < \varepsilon$, which implies $q_i^n \,\tilde{\in}\, B(q\varepsilon)$ and $q_i^{nc} \,\tilde{\in}\, B(q^c\varepsilon)$. Therefore (X, d) is separable. Q.E.D.

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